Pattern formation in nonequilibrium physics

J. P. Gollub

Haverford College, Haverford, Pennsylvania 19041
and Department of Physics, University of Pennsylvania,
Philadelphia, Pennsylvania 19104

J. S. Langer

Department of Physics, University of California, Santa Barbara, California 93106

Remarkable and varied pattern-forming phenomena occur in fluids and in phase transformations. The authors describe and compare some of these phenomena, offer reflections on their similarities and differences, and consider possibilities for the future development of this field.

[S0034-6861(99)04702-9]

I. INTRODUCTION

The complex patterns that appear everywhere in nature have been cause for wonder and fascination throughout human history. People have long been puzzled, for example, about how intricate snowflakes can form, literally, out of thin air; and our minds boggle at the elegance of even the simplest living systems. As physicists, we have learned much about natural pattern formation in recent years; we have discovered how rich this subject can be, and how very much remains to be understood. Our growing understanding of the physics of pattern formation has led us to speculate—so far with only limited success—about a more general science of complexity, and to pose deep questions about our ability to predict and control natural phenomena.

Although pattern formation—i.e., morphogenesis—has always been a central theme in natural philosophy, it has reemerged in mainstream nonequilibrium physics only in the last quarter of the 20th Century. This has happened, in part, as an outgrowth of physicists' and materials scientists' interest in phase transitions. Many of the most familiar examples of pattern formation occur in situations in which a system is changing from one phase to another—from a liquid to a geometrically patterned solid, for example, or from a uniform mixture of chemical constituents to a phase-separated pattern of precipitates. As scientists have learned more about the equilibrium aspects of phase transitions, many have become interested in the non-equilibrium processes that accompany them. This line of investigation has led directly to questions of pattern formation.

Another direction from which physicists have approached the study of pattern formation has been the theory of nonlinear dynamical systems. Mechanical systems that can be described by ordinary differential equations often undergo changes from simple to complex behavior in response to changes in their control parameters. For example, the periodically forced and damped pendulum shows chaotic motion for certain intervals of the forcing amplitude, as well as periodic windows within the chaotic domains—a temporal "pattern" with considerable complexity. This is a simple case, however, with only a few degrees of freedom. More relevant for the present purposes are spatially extended dynamical systems with many degrees of freedom, for which partial differential equations are needed. Corresponding physical systems include fluids subjected to heating or rotation, which exhibit sequences of increasingly complex spatiotemporal patterns as the driving forces change. These are all purely deterministic pattern-forming systems. Understanding how they behave has been a crucial step toward understanding deterministic chaos—one of the most intriguing and profound scientific concepts to emerge in this century.

At the center of our modern understanding of pattern formation is the concept of instability. It is interesting to note that the mathematical description of instabilities is strikingly similar to the phenomenological theory of phase transitions first given by Landau (Landau and Lifshitz, 1969). We now know that complex spatial or temporal patterns emerge when relatively simple systems are driven into unstable states, that is, into states that will deform by large amounts in response to infinitesimally small perturbations. For example, solar heating of the earth's surface can drive Rayleigh-Bénard-like convective instabilities in the lower layer of the atmosphere, and the resulting flow patterns produce fairly regular arrays of clouds. At stronger driving forces, the convection patterns become unstable and turbulence increases. Another familiar example is the roughness of fracture surfaces produced by rapidly moving cracks in brittle solids. When we look in detail, we see that a straight crack, driven to high enough speeds, becomes unstable in such a way that it bends, sends out sidebranching cracks, and produces damage in the neighboring material. In this case, the physics of the instability that leads to these irregular patterns is not yet known.

After an instability has produced a growing disturbance in a spatially uniform system, the crucial next step in the pattern-forming process must be some intrinsically nonlinear mechanism by which the system moves toward a new state. That state may resemble the unstable deformation of the original state—the convective rolls in the atmosphere have roughly the same spacing as the wavelength of the initial instability. However, in many other cases, such as the growth of snowflakes, the new patterns look nothing like the linearly unstable deformations from which they started. The system evolves in entirely new directions as determined by nonlinear dynamics. We now understand that it is here, in the nonlinear phase of the process, that the greatest scientific challenges arise.

The inherent difficulty of the pattern-selection problem is a direct consequence of the underlying (linear or nonlinear) instabilities of the systems in which these phenomena occur. A system that is linearly unstable is one for which some response function diverges. This means that pattern-forming behavior is likely to be extremely sensitive to small perturbations or small changes in system parameters. For example, many patterns that we see in nature, such as snowflake-like dendrites in solidifying alloys, are generated by selective amplification of atomic-scale thermal noise. The shapes and speeds of growing dendrites are also exquisitely sensitive to tiny crystalline anisotropies of surface energies.

Some important questions, therefore, are: Which perturbations and parameters are the sensitively controlling ones? What are the mechanisms by which those small effects govern the dynamics of pattern formation? What are the interrelations between physics at different length scales in pattern-forming systems? When and how do atomic-scale mechanisms control macroscopic phenomena? At present, we have no general strategy for answering these questions. The best we have been able to do is to treat each case separately and—because of the remarkable complexity that has emerged in many of these problems—with great care.

In the next several sections of this article we discuss the connection between pattern formation and nonlinear dynamics, and then describe just a few specific examples that illustrate the roles of instability and sensitivity in nonequilibrium pattern formation. Our examples are drawn from fluid dynamics, granular materials, and crystal growth, which are topics that we happen to know well. We conclude with some brief remarks, mostly in the form of questions, about universality, predictability, and long-term prospects for this field of research.

II. PATTERN FORMATION AND DYNAMICAL SYSTEMS

Our understanding of pattern formation has been dramatically affected by developments in mathematics. Deterministic pattern-forming systems are generally described by nonlinear partial differential equations, for example, the Navier-Stokes equations for fluids, or reaction-diffusion equations for chemical systems. It is characteristic of such nonlinear equations that they can have multiple steady solutions for a single set of control parameters such as external driving forces or boundary conditions. These solutions might be homogeneous, or patterned, or even more complex. As the control parameters change, the solutions appear and disappear and change their stabilities. In mathematical models of spatially extended systems, different steady solutions can coexist in contact with each other, separated by lines of defects or moving fronts.

The best way to visualize the solutions of such equations is to think of them as points in a multidimensional mathematical space spanned by the dynamical variables, that is, a "phase space." The rules that determine how these points move in the phase space constitute what we call a "dynamical system." One of the most important developments in this field has been the recognition that dynamical systems with infinitely many degrees of freedom can often be described by a finite number of relevant variables, that is, in finite-dimensional phase spaces. For example, the flow field for Rayleigh-Bénard convection not too far from threshold can be described accurately by just a few time-dependent Fourier amplitudes. If we think of the partial differential equations as being equivalent to finite sets of coupled ordinary differential equations, then we can bring powerful mathematical concepts to bear on the analysis of their solutions.

As we shall emphasize in the next several sections of this article, dynamical-systems theory provides at best a qualitative framework on which to build physical models of pattern formation. Nevertheless, it has produced valuable insights and, in some cases, has even led to prediction of novel effects. It will be useful, therefore, to summarize some of these general concepts before looking in more detail at specific examples. An introductory discussion of the role of dynamical systems theory in fluid mechanics has been given by Aref and Gollub (1996).

In dynamical-systems theory, the stable steady solutions of the equations of motion are known as "stable fixed points" or "attractors," and the set of points in the phase space from which trajectories flow to a given fixed point is its "basin of attraction." As the control parameters are varied, the system typically passes through "bifurcations" in which a fixed point loses its stability and, at the same time, one or more new stable attractors appear. An especially simple example is the "pitchfork" bifurcation at which a stable fixed point representing a steady fluid flow, for example, gives rise to two symmetry-related fixed points describing cellular flows with opposite polarity. Many other types of bifurcation have been identified in simple models and also have been seen in experiments.

The theory of bifurcations in dynamical systems helps us understand why it is sometimes reasonable to describe a system with infinitely many degrees of freedom using only a finite (or even relatively small) number of dynamical variables. An important mathematical result known as the "center manifold theorem" (Guckenheimer and Holmes, 1983) indicates that, when a bifurcation occurs, the associated unstable trajectories typically move away from the originally stable fixed point only within a low-dimensional subspace of the full phase space. The subspace is "attracting" in the sense that trajectories starting elsewhere converge to it, so that the degrees of freedom outside the attracting subspace are effectively irrelevant. It is for this reason that we may need only a low-dimensional space of dynamical variables to describe some pattern-formation problems near their thresholds of instability—a remarkable physical result.

Time-varying states, such as oscillatory or turbulent flows, are more complex than simple fixed points. Here, some insight also has been gained from considering dynamical systems. Oscillatory behavior is generally described as a flow on a limit cycle (or closed loop) in phase space, and chaotic states may be represented by more complex sets called "strange attractors." The most characteristic feature of the latter may be understood in terms of the Lyapunov exponents that give the local exponential divergence or convergence rates between two nearby trajectories, in the different directions along and transverse to those trajectories. If at least one of these exponents, when averaged over time, is positive, then nearby orbits will separate from each other exponentially in time. Provided that the entire attracting set is bounded, the only possibility is for the set to be fractal.

In the early 1970s, strange attractors were thought by some to be useful models for turbulent fluids. In fact, the phase-space paradigm can give only a caricature of the real physics because of the large number of relevant degrees of freedom involved in most turbulent flows. While much less than $6N$ (the number of degrees of freedom for a system of N molecules), that number still grows in proportion to $R^{3/4}$, where R is the Reynolds number. We do not yet know whether weakly turbulent states ("spatiotemporal chaos") that sometimes occur near the onset of instability may be viewed usefully using the concepts of dynamical systems.

III. PATTERNS AND SPATIOTEMPORAL CHAOS IN FLUIDS: NONLINEAR WAVES

Pattern formation has been investigated in an immense variety of hydrodynamic systems. Examples include convection in pure fluids and mixtures; rotating fluids, sometimes in combination with thermal transport; nonlinear surface waves at interfaces; liquid crystals driven either thermally or by electromagnetic fields; chemically reacting fluids; and falling droplets. Some similar phenomena occur in nonlinear optics. Many of these cases have been reviewed by Cross and Hohenberg (1993); there also have been a host of more recent developments. Since it is not possible in a brief space to discuss this wide range of phenomena, we focus here on an example that poses interesting questions about the nature of pattern formation: waves on the surfaces of fluids. We shall also make briefer remarks about other fluid systems that have revealed strikingly novel phenomena.

The surface of a fluid is an extended dynamical system for which the natural variables are the amplitudes and phases of the wavelike deformations. These waves were at one time regarded as being essentially linear at small amplitudes. However, even weak nonlinear effects cause interactions between waves with different wave vectors and can be important in determining wave patterns. When the wave amplitudes are large, the nonlinear effects lead to chaotic dynamics in which many degrees of freedom are active.

A convenient way of exciting nonlinear waves in a manner that does not directly break any spatial symmetry is to subject the fluid container to a small-amplitude vertical excitation. This leads to standing waves at half the driving frequency, via an instability and an associated bifurcation first demonstrated by Faraday (1831), 68 years before the American Physical Society was founded. The characteristic periodicity of the resulting patterns is approximately the same as the wavelength of the most rapidly growing linear instability determined by the dispersion relation for capillary-gravity waves, but the wave patterns themselves are far more complex and interesting.

All of the regular patterns that can tile the plane have been found in this system, including hexagons, squares, and stripes (Kudrolli and Gollub, 1996). In addition, various types of defects that are analogous to crystalline defects occur: grain boundaries, dislocations, and the like. Which patterns are stable depends on parameters: the fluid viscosity, the driving frequency, and the acceleration. Significant domains of coexistence between different patterns are also known, where patterns with different symmetry are simultaneously stable.

These phenomena have resisted quantitative explanation for a number of reasons: the difficulty of dealing with boundary conditions at the moving surface of the fluid; the nonlinearity of the hydrodynamic equations; and the complex effects of viscosity. However, a suitable mathematical description, consistent with the general framework of dynamical-systems theory, is now available (Chen and Viñals, 1997), and it leads to a satisfactory explanation of these pattern-forming phenomena. The basic idea is to regard the surface as a superposition of interacting waves propagating in different directions. Coupled evolution equations can be written for the various wave amplitudes. The coupling coefficients depend on the angles between the wave vectors, and these coupling functions depend in turn on the imposed parameters (such as wave frequency). The entire problem is variational, but only near the threshold of wave formation. That is, the preferred pattern near the onset of instability is the one that minimizes a certain functional of the wave amplitudes, in much the same way that the preferred state of a crystal is the one that minimizes its free energy. Away from threshold, on the other hand, no such variational principle exists, and the variety of behaviors is correspondingly richer.

These results raise the question of whether other types of regular patterns can be formed that are not spatially periodic but do have rotational symmetry, i.e., quasicrystalline patterns. In fact they do occur (Christiansen *et al.*, 1992; Edwards and Fauve, 1994), just as they do in ordinary crystals. The way in which these different patterns become stable or unstable as the parameters are varied has now been worked out in some detail and appears to be in accord with experiment (Binks and van de Water, 1997). An example of a quasicrystalline pattern is shown in Fig. 1.

When the wave amplitudes are raised sufficiently, transitions to spatially and temporally disordered states

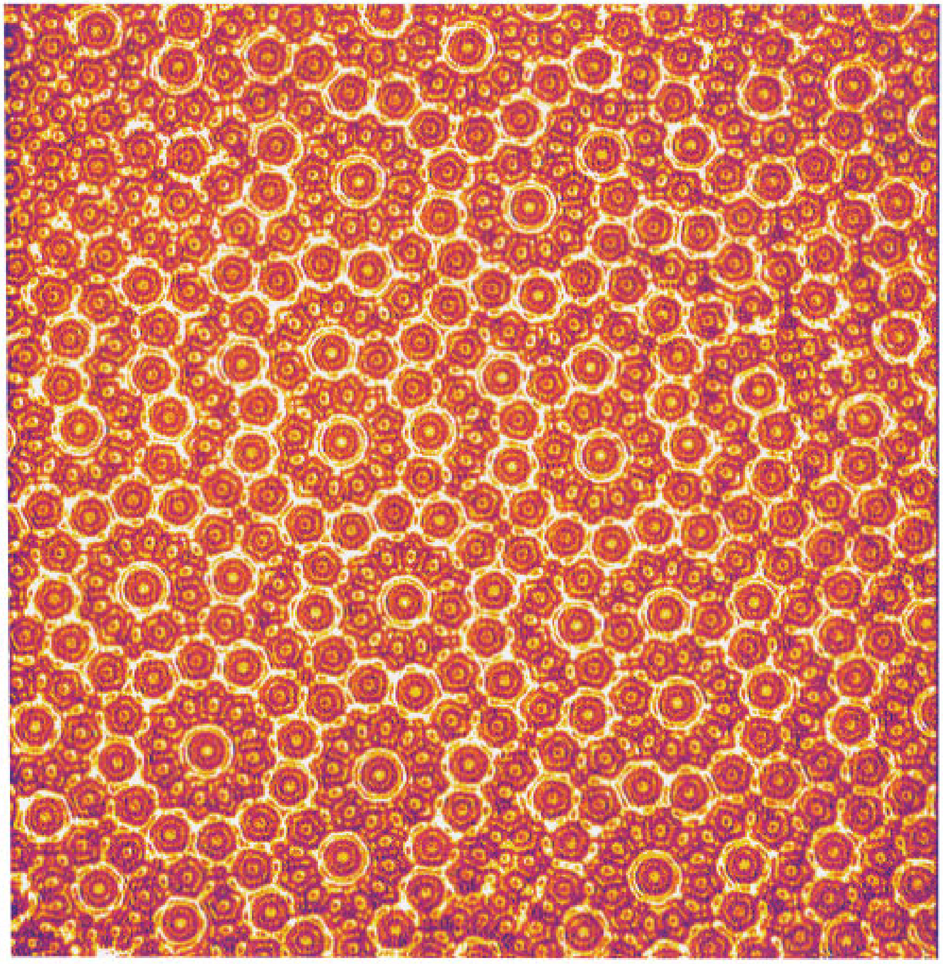

FIG. 1. (Color) A quasicrystalline wave pattern with 12-fold rotational symmetry. This standing-wave pattern was produced by forcing a layer of silicone oil simultaneously at two frequencies, using a method invented by Edwards and Fauve. The brightest regions are locally horizontal, whereas darker colors indicate inclined regions. From work done at Haverford for an undergraduate thesis by B. Pier.

occur (Kudrolli and Gollub, 1996 and references therein). In the language of dynamical systems, some of these new states might be called ''strange attractors,'' although they are certainly not low-dimensional objects. They are much less well understood than the standing-wave states, and the ways in which they form appears to depend on the ordered states from which they emerge. For example, the hexagonal lattice appears to melt continuously, while the striped phase breaks down inhomogeneously in regions where the stripes are most strongly curved. The resulting states of spatiotemporal chaos are not completely disordered; there can be regions of local order. Furthermore, if the fluid is not too viscous, so that the correlation length of the pattern is relatively long, then the symmetry imposed by the boundaries can be recovered by averaging over a large number of individually fluctuating patterns (Gluckman *et al.*, 1995). A case of strongly turbulent capillary waves has also been studied experimentally (Wright *et al.*, 1997).

Certain other fluid systems have chaotic states that occur closer to the linear threshold of the primary pattern. In these cases, quantitative comparison with theory is sometimes possible. An example is the behavior of Rayleigh-Bénard convection in the presence of rotation about a vertical axis (Hu *et al.*, 1997). This problem is relevant to atmospheric dynamics. Though the basic pattern consists of rolls, as shown in Fig. 2, they are unstable. Patches of rolls at different angles invade each other as time proceeds, and the pattern remains time dependent indefinitely. This phenomenon has been discussed theoretically (Tu and Cross, 1992) using a two-dimensional nonlinear partial differential equation known as the complex Ginzburg-Landau equation. Similar models have been used successfully for treating a variety of nonchaotic pattern-forming phenomena. In this case, the model is able to reproduce the qualitative behavior of the experiments, but does not successfully describe the divergence of the correlation length of the patchy chaotic fluctuations as the transition is approached. Thus the goal of understanding spatiotemporal chaos has remained elusive.

There is one area where the macroscopic treatment of pattern-forming instabilities connects directly to microscopic physics: the effects of thermal noise. Macroscopic patterns often emerge from the amplification of noise by instabilities. Therefore, fluctuations induced by thermal

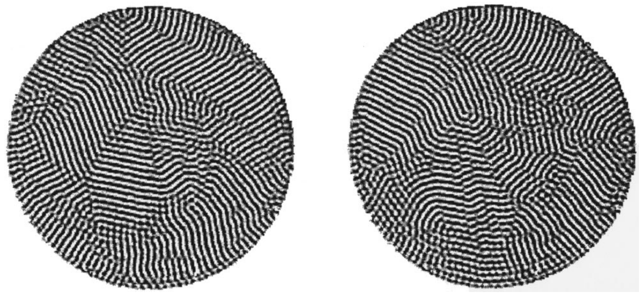

FIG. 2. Spatiotemporal chaos in rotating Rayleigh-Bénard convection shown at two different times. Patches of rolls at different angles invade each other as time proceeds. Courtesy of G. Ahlers.

noise (as distinguished from chaotic fluctuations produced by nonlinearity) should be observable near the threshold of instability. This remarkable effect has been demonstrated quantitatively in several fluid systems, for example, ordinary Rayleigh-Bénard convection (Wu *et al.*, 1995). As we shall see in Sec. V, very similar amplification of thermal noise occurs in dendritic crystal growth.

IV. PATTERNS IN GRANULAR MATERIALS

Patterns quite similar to the interfacial waves described in the previous section occur when the fluid is replaced by a layer of granular matter such as sand or, in well-controlled recent experiments, uniform metallic or glassy spheres (Melo *et al.*, 1995). Depending on the frequency and amplitude of the oscillation of the container, the upper surface of the grains can arrange itself into arrays of stripes or hexagons, as shown in Fig. 3. Lines dividing regions differing in their phase of oscillation, and disordered patterns, are also evident.

In addition, granular materials can exhibit localized solitary excitations known as ‘‘oscillons’’ (Umbanhowar *et al.*, 1996). These can in turn organize themselves into clusters, as shown in Fig. 3(d). This striking discovery has given rise to a number of competing theories and has been immensely provocative. It is interesting to note that localized excitations are also found in fluids. For example, they have been detected in instabilities induced by electric fields applied across a layer of nematic liquid crystal (Dennin *et al.*, 1996). All of these localized states are intrinsically nonlinear phenomena, whether they occur in granular materials or in ordinary fluids, and do not resemble any known linear instability of the uniform system.

Granular materials have been studied empirically for centuries in civil engineering and geology. Nevertheless, we still have no fundamental physical understanding of their nonequilibrium properties. In fact, to a modern physicist, granular materials look like a novel state of matter. For a review of this field, see Jaeger *et al.* (1996), and references therein.

There are several clear distinctions between granular materials and other, superficially comparable, many-body systems such as fluids. Because they have huge numbers of degrees of freedom, they can only be understood in statistical terms. However, individual grains of sand are enormously more massive than atoms or even macromolecules; thus thermal kinetic energy is irrelevant to them. On the other hand, each individual grain has an effectively infinite number of internal degrees of freedom; thus the grains are generally inelastic in their interactions with each other or with boundaries. They also may have irregular shapes; arrays of such grains may achieve mechanical equilibrium in a variety of configurations and packings. It seems possible, therefore, that concepts like ‘‘temperature’’ and ‘‘entropy’’ might be useful for understanding the behavior of these materials (for example, see Campbell, 1990).

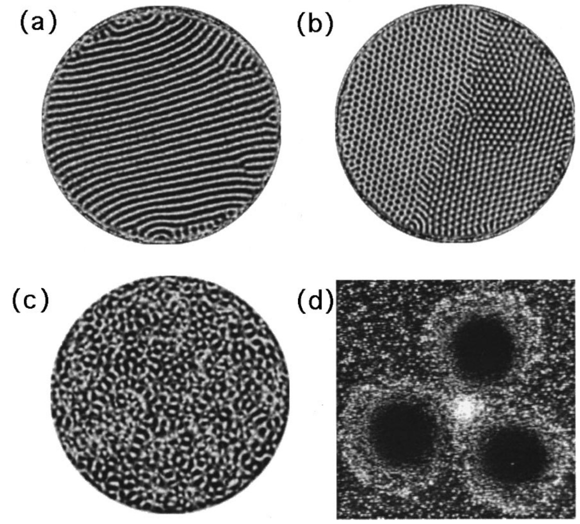

FIG. 3. Standing-wave patterns in a vibrating layer of granular material. (a) Stripes; (b) hexagons and defects; (c) disordered waves; (d) clusters of localized ‘‘oscillons.’’ Courtesy of P. Umbanhowar and H. Swinney.

In the oscillating-granular-layer experiments, some of the simpler transitions can be explained in terms of temporal symmetry breaking resulting from the the low-dimensional dynamics of the particles as they bounce off the oscillating container surface. The onset of temporal period doubling in the particle dynamics coincides with the spatial transition from stripes to hexagons. Both the particle trajectories and the spatial patterns are then different on successive cycles of the driver. The analogous hexagonal state for Faraday waves in fluids can be induced either by temporal symmetry breaking of the external forcing, or by the frequency and viscosity dependence of the coupling between different traveling-wave components. Both of these mechanisms are quite different microscopically from the single-particle dynamics that generates the hexagons in granular materials.

On the other hand, there are substantial similarities between the granular and fluid behaviors. The granular material expands or dilates as a result of excitation. Roughly speaking, dilation of the granular layer reduces the geometrical constraints that limit flow. This corresponds to lower viscosity of the conventional fluid. Dilation accounts in physical terms for the fact that the striped phase that occurs at high fluid viscosity may be found at low acceleration of the granular material.

The differences, however, seem to be emerging dramatically as new experiments and numerical simulations probe more deeply into granular phenomena. For example, stresses in nearly static granular materials are highly inhomogeneous, forming localized stress chains that are quite unlike anything seen in ordinary liquids and solids. Even in situations involving flow, the behavior of granular materials often seems to be governed by their tendency to ‘‘jam,’’ that is, to get themselves into local configurations from which they are temporarily unable to escape. This happens during a part of each cycle in the oscillating-layer experiments. (The concept of ‘‘jammed’’ systems was the topic of a Fall 1997 program at the Institute for Theoretical Physics in Santa Barbara. For more information, consult the ITP web site: http://www.itp.ucsb.edu/online/jamming2/schedule.html.)

V. GROWTH AT INTERFACES: DENDRITIC SOLIDIFICATION

We turn finally to the topic of dendritic pattern formation. It is here that some of the deepest questions in this field—the mathematical subtlety of the selection problem and the sensitivity to small perturbations—have emerged most clearly in recent research.

Dendritic solidification, that is, the ‘‘snowflake problem,’’ is one of the most thoroughly investigated topics in the general area of nonequilibrium pattern formation. It is only in the last few years, however, that we finally have learned how these elegant dendritic crystals are formed in the atmosphere, and why they occur with such diversity that no two of them ever seem to be exactly alike. Nevertheless, our present understanding is still far from good enough for many practical purposes, for example, for predicting the microstructures of multicomponent cast alloys.

Much of the research on dendritic crystal growth has been driven, not only by our natural curiosity about such phenomena, but also by the need to understand and control metallurgical microstructures. (For example, see Kurz and Fisher, 1989) The interior of a grain of a freshly solidified alloy, when viewed under a microscope, often looks like an interlocking network of highly developed snowflakes. Each grain is formed by a dendritic, i.e., treelike, process in which a crystal of the primary composition grows out rapidly in a cascade of branches and sidebranches, leaving solute-rich melt to solidify more slowly in the interstices. The speed at which the dendrites grow and the regularity and spacing of their sidebranches determine the observed microstructure which, in turn, governs many of the properties of the solidified material such as its mechanical strength and its response to heating and deformation.

The starting point for investigations of metallurgical microstructures or snowflakes is the study of single, isolated, freely growing dendrites. Remarkable progress has been made on understanding this phenomenon recently. The free-dendrite problem is most easily defined by reference to the xenon dendrite shown in Fig. 4. Here, we are looking at a pure single crystal growing into its liquid phase. The speed at which the tip is advancing, the radius of curvature of the tip, and the way in which the sidebranches emerge behind the tip, all are determined uniquely by the degree of undercooling, i.e., by the degree to which the liquid is colder than its freezing temperature. The question is: How?

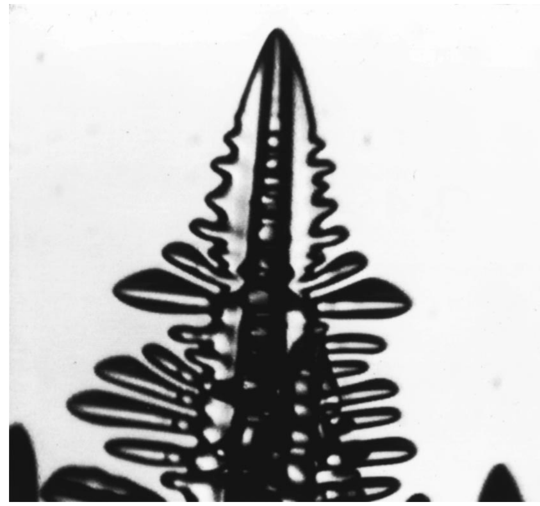

FIG. 4. Dendritic xenon crystal growing in a supercooled melt. Courtesy of J. Bilgram.

In the most common situations, dendritic growth is controlled by diffusion—either the diffusion of latent heat away from the growing solidification front or the diffusion of chemical constituents toward and away from that front. These diffusion effects very often lead to shape instabilities; small bumps grow out into fingers because, like lightning rods, they concentrate the diffusive fluxes ahead of them and therefore grow out more rapidly than a flat surface. This instability, generally known as the ‘‘Mullins-Sekerka instability,’’ is the trigger for pattern formation in solidification.

Today’s prevailing theory of free dendrites is generally known as the ‘‘solvability theory’’ because it relates the determination of dendritic behavior to the question of whether or not there exists a sensible solution for a certain diffusion-related equation that contains a singular perturbation. The term ‘‘singular’’ means that the perturbation, in this case the surface tension at the solidification front, completely changes the mathematical nature of the problem whenever it appears, no matter how infinitesimally weak it might be. In the language of dynamical systems, the perturbation controls whether or not there exists a stable fixed point. Similar situations occur in fluid dynamics, for example, in the ‘‘viscous fingering’’ problem, where a mechanism similar to the Mullin-Sekerka instability destabilizes a moving interface between fluids of different viscosities, and a solvability mechanism determines the resulting fingerlike pattern. (See Langer, 1987, for a pedagogical introduction to solvability theory, and Langer, 1989, for an overview including the viscous fingering problem.)

The solvability theory has been worked out in detail for many relevant situations such as the xenon dendrite shown in Fig. 4. (See Bisang and Bilgram, 1996, for an account of the xenon experiments, and also for references to recent theoretical work by Brener and colleagues.) The theory predicts how pattern selection is determined, not just by the surface tension (itself a very small correction in the diffusion equations), but by the crystalline anisotropy of the surface tension—an even weaker perturbation in this case. It further predicts that the sidebranches are produced by secondary instabilities near the tip that are triggered by thermal noise and amplified in special ways as they grow out along the sides of the primary dendrite. The latter prediction is especially remarkable because it relates macroscopic features—sidebranches with spacings of order tens of microns—to molecular fluctuations whose characteristic sizes are of order nanometers.

Each of those predictions has been tested in the xenon experiment, quantitatively and with no adjustable fitting parameters. They have also been checked in less detail in experiments using other metallurgical analog materials. In addition, the theory has been checked in numerical studies that have probed its nontrivial mathematical aspects (Karma and Rappel, 1996). As a result, although we know that there must be other cases (competing thermal and chemical effects, for example, or cases where the anisotropy is large enough that it induces faceting), we now have reason for confidence that we understand at least some of the basic principles correctly.

VI. REFLECTIONS AND CONCLUSIONS

We have illustrated pattern-forming phenomena through a few selected examples, each of which has given rise to a large literature. Many others could be cited. As we have seen, the inherent sensitivity of pattern-forming mechanisms to small perturbations means that research in this field must take into account physical phenomena across extraordinarily wide ranges of length and time scales. Moreover, studies of pattern formation increasingly are being extended to materials that are more complex than isotropic classical fluids and homogeneous solids. The case of granular matter described here is one example of this trend. An important example for the future is pattern formation in biological systems, where the interplay between physical effects and genetic coding leads to striking diversity.

The expanding complexity and importance of this field brings urgency to a set of deep questions about theories of pattern formation and, more generally, about the foundations of nonequilibrium statistical physics. What does sensitivity to noise and delicate perturbations imply about the apparent similarities between different systems? Are there, for example, deep connections between dendritic sidebranching and fracture, or are the apparent similarities superficial and unimportant? What about the apparent similarities between the patterns seen in fluids and granular materials? In short, are there useful "universality classes" for which detailed underlying mechanisms are less important than, for example, more general symmetries or conservation laws? Might we discover some practical guidelines to tell us how to construct predictive models of pattern-forming systems, or shall we have to start from the beginning in considering each problem?

These are not purely philosophical questions. Essentially all processes for manufacturing industrial materials are nonequilibrium phenomena. Most involve, at one stage or another, some version of pattern formation. The degree to which we can develop quantitative, predictive models of these phenomena will determine the degree to which we can control them and perhaps develop entirely new technologies. Will we be able, for example, to write computer programs to predict and control the microstructures that form during the casting of high-performance alloys? Can we hope to predict, long in advance, mechanical failure of complex structural materials? Will we ever be able to predict earthquakes? Or, conversely, might we discover that the complexity of many systems imposes intrinsic limits to our ability to predict their behavior? That too would be an interesting and very important outcome of research in this field.

ACKNOWLEDGMENTS

J.S.L. acknowledges support from DOE Grant DE-FG03-84ER45108. JPG acknowledges support from NSF Grant DMR-9704301.

REFERENCES

Aref, H., and J.P. Gollub, 1996, in *Research Trends in Fluid Dynamics: Report From the United States National Committee on Theoretical and Applied Mechanics,* edited by J.L. Lumley (American Institute of Physics, Woodbury, NY), p. 15.

Binks, D., and W. van de Water, 1997, Phys. Rev. Lett. **78**, 4043.

Bisang, U., and J.H. Bilgram, 1996, Phys. Rev. E **54**, 5309.

Campbell, C.S., 1990, Annu. Rev. Fluid Mech. **22**, 57.

Chen, P., and J. Viñals, 1997, Phys. Rev. Lett. **79**, 2670.

Christiansen, B., P. Alstrom, and M.T. Levinsen, 1992, Phys. Rev. Lett. **68**, 2157.

Cross, M.C., and P.C. Hohenberg, 1993, Rev. Mod. Phys. **65**, 851.

Dennin, M., G. Ahlers, and D.S. Cannell, 1996, Phys. Rev. Lett. **77**, 2475.

Edwards, S., and S. Fauve, 1994, J. Fluid Mech. **278**, 123.

Faraday, M., 1831, Philos. Trans. R. Soc. London **121**, 299.

Gluckman, B.J., C.B. Arnold, and J.P. Gollub, 1995, Phys. Rev. E **51**, 1128.

Guckenheimer, J., and P. Holmes, 1983, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York).

Hu, Y., R.E. Ecke, and G. Ahlers, 1997, Phys. Rev. E **55**, 6928.

Jaeger, H.M., S.R. Nagel, and R.P Behringer, 1996, Rev. Mod. Phys. **68**, 1259.

Karma, A., and J. Rappel, 1996, Phys. Rev. Lett. **77**, 4050.

Kudrolli, A., and J.P. Gollub, 1996, Physica D **97**, 133.

Kurz, W., and D.J. Fisher, 1989, *Fundamentals of Solidification* (Trans Tech Publications, Brookfield, VT).

Landau, L.D., and E.M. Lifshitz, 1969, *Statistical Physics* (Pergamon, Oxford), Chap. XIV.

Langer, J.S., 1987, in *Chance and Matter, Proceedings of the Les Houches Summer School, Session XLVI*, edited by J. Souletie, J. Vannimenus, and R. Stora (North Holland, Amsterdam), p. 629.

Langer, J.S., 1989, Science **243**, 1150.

Melo, F., P.B. Umbanhowar, and H.L. Swinney, 1995, Phys. Rev. Lett. **75**, 3838.

Miles, J., and D. Henderson, 1990, Annu. Rev. Fluid Mech. **22**, 143.

Tu, Y., and M. Cross, 1992, Phys. Rev. Lett. **69**, 2515.

Umbanhowar, P.B., F. Melo, and H.L. Swinney, 1996, Nature (London) **382**, 793.

Wright, W.B., R. Budakian, D.J. Pine, and S.J. Putterman, 1997, Science **278**, 1609.

Wu, M., G. Ahlers, and D.S. Cannell, 1995, Phys. Rev. Lett. **75**, 1743.